REPRESENTABLE MONOTONE OPERATORS AND LIMITS OF SEQUENCES OF MAXIMAL MONOTONE OPERATORS

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Abstract. We show that the lower limit of a sequence of maximal monotone operators on a reflexive Banach space is a representable monotone operator. As a consequence, we obtain that the variational sum of maximal monotone operators and the variational composition of a maximal monotone operator with a linear continuous operator are both representable monotone operators.

1. Introduction

In recent years, the utilization of Fitzpatrick functions in the study of monotone operators gave rise to a new class of monotone operators, the so-called *representable* operators. These are operators $T : X \rightrightarrows X^*$ from a Banach space X into its dual X^* whose graphs can be described via a convex lower semicontinuous function f defined on $X \times X^*$, namely: $T = \{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x^*, x\rangle\}$. Maximal monotone operators are representable, but not all representable operators are maximal monotone. However, it turns out that representable operators possess properties similar to those of maximal monotone operators, and that representable operators with full-space domain are actually maximal monotone (see Section 2). Thus, in situations where maximal monotonicity is lacking or not known, it is interesting to know whether representability is present. Three such situations are studied in this paper: the lower limit of sequences of maximal monotone operators (Section 3), the variational sum of two maximal monotone operators (Section 4) and the variational composition of a maximal monotone operator with a linear continuous map (Section 5).

Let $T_n : X \rightrightarrows X^*$ be a sequence of maximal monotone operators between a reflexive strictly convex Banach space X and its strictly convex dual X^*. It is known that the strong (graph-)lower limit of such a sequence, denoted $\liminf T_n$, is monotone but not maximal monotone in general. Here we show that it is however representable (Theorem 3.3). On the other hand, it is known that in finite-dimensional spaces, the so-called Painlevé-Kuratowski (graph-)limit of the T_n's is indeed maximal monotone (Attouch's theorem [2, 3]). We extend this theorem to the case of arbitrary reflexive strictly convex Banach spaces, replacing, as usual, Painlevé-Kuratowski convergence by Mosco convergence (Theorem 3.5).

Next, we consider the *variational sum* of two maximal monotone operators T_1, T_2. This concept was introduced by Attouch-Baillon-Théra [3] in Hilbert spaces as a substitute for the usual (Minkowski) sum $T_1 + T_2$ which in general does not yield a maximal monotone operator. It is given by

$$T_1 \underset{v}{+} T_2 := \bigcap_{\mathcal{I}} \liminf_n \,(T_{1,\lambda_n} + T_{2,\mu_n}),$$

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where $\mathcal{I} = \Big\{\{(\lambda_n, \mu_n)\} \subset \mathbb{R}^2 : \lambda_n,\, \mu_n \geq 0,\, \lambda_n + \mu_n > 0, \lambda_n, \mu_n \to 0\Big\}$, and where T_{1,λ_n} and T_{2,μ_n} denote the Yosida regularizations of T_1 and T_2 respectively. The operators $T_n = T_{1,\lambda_n} + T_{2,\mu_n}$ are maximal monotone, so as a consequence of our previous result on the lower limit of sequences of maximal monotone operators, we derive that the variational sum is actually a representable extension of the usual sum $T_1 + T_2$.

Analog results for *variational composition* are also obtained in the last section.

2. Maximal monotone and representable operators

Set-valued mappings $T : X \rightrightarrows Y$ between sets X and Y are identified with their graphs $T \subset X \times Y$, so $x^* \in Tx$ is equally written as $(x, x^*) \in T$. The *values* of $T : X \rightrightarrows Y$ are the subsets $Tx \subset Y$ for $x \in X$, the *inverse* of T is the mapping $T^{-1} : Y \rightrightarrows X$ defined by $T^{-1}y = \big\{x \in X : y \in Tx\big\}$, and the domain of T is the projection of (the graph of) T onto X, that is, $\operatorname{Dom} T = \{x \in X : Tx \neq \emptyset\}$.

In what follows, X denotes a Banach space, X^* its continuous dual, B_{X^*} the unit ball in X^*, and $X \times X^*$ is equipped with the strong$\times$weak-star ($s \times w^*$) topology. Recall that a set-valued operator $T : X \rightrightarrows X^*$, or a subset $T \subset X \times X^*$, is said to be

• *monotone* provided $\langle y^* - x^*, y - x\rangle \geq 0,\ \forall (x, x^*), (y, y^*) \in T$,

• *maximal monotone* provided it is monotone and maximal (under set inclusion) in the family of all monotone sets contained in $X \times X^*$,

• *representable* provided there is a lower semicontinuous convex function $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that

$$\begin{cases} f(x, x^*) \geq \langle x^*, x\rangle, & \forall\, (x, x^*) \in X \times X^*, \\ f(x, x^*) = \langle x^*, x\rangle \Leftrightarrow (x, x^*) \in T. \end{cases}$$

Every representable operator is indeed monotone (see, e.g., Penot-Zălinescu [13]).

Using the notations of Martínez-Legaz-Svaiter [10],

$$\mathcal{F} = \{f : X \times X^* \to \mathbb{R} \cup \{+\infty\} \text{ lower semicontinuous convex} : f(x, x^*) \geq \langle x^*, x\rangle\ \forall\, (x, x^*)\},$$

and, given $f \in \mathcal{F}$,

$$L(f) = \{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x^*, x\rangle\},$$

we can write more synthetically:

$$T \text{ representable } \iff \exists f \in \mathcal{F} \,:\, T = L(f). \tag{2.1}$$

For a nonempty $T : X \rightrightarrows X^*$, consider $\varphi_T, \varphi_T^* : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ given by

$$\begin{cases} \varphi_T(x, x^*) = \sup\{\, \langle y^*, x\rangle - \langle y^*, y\rangle + \langle x^*, y\rangle : (y, y^*) \in T\,\}, \\ \varphi_T^*(x, x^*) = \sup\{\, \langle (y, y^*), (x, x^*)\rangle - \varphi_T(y, y^*) : (y, y^*) \in X \times X^*\,\}. \end{cases}$$

Obviously, φ_T and φ_T^* are convex and lower semicontinuous in $X \times X^*$ supplied with the strong$\times$weak-star topology. Both functions were considered by Fitzpatrick [7]. Since then, they have been recognized to be quite useful in the study of monotone operators (see, e.g., [4, 5, 6, 10, 12, 19, 20, 21, 22, 23]). The following proposition shows examples of classifications of monotone operators using these functions.

Proposition 2.1 (see, e.g., [10, 12, 22])**.** *Let X be a Banach space and let $T : X \rightrightarrows X^*$ with $\operatorname{Dom} T \neq \emptyset$. Then:*

(1) *T monotone $\iff \varphi_T^* \in \mathcal{F}$ and $T \subset L(\varphi_T^*)$.*

(2) *T representable* $\iff \varphi_T^* \in \mathcal{F}$ *and* $T = L(\varphi_T^*)$.
(3) *T maximal monotone* $\iff \varphi_T \in \mathcal{F}$ *and* $T = L(\varphi_T^*)$.

Example. The operator $T = \{(0,0)\} \subset \mathbb{R} \times \mathbb{R}$ (considered in Fitzpatrick [7]) is representable but not maximal monotone. Easy computation shows that $\varphi_T(x, x^*) = 0$ for every (x, x^*), while $\varphi_T^*(x, x^*) = \delta_T(x, x^*)$ for every (x, x^*), where δ_T, the *indicator function* of (the graph of) T, is equal to 0 on T, to $+\infty$ outside. Hence $\varphi_T \notin \mathcal{F}$, $\varphi_T^* \in \mathcal{F}$, and $T = L(\varphi_T^*)$. Any linear map is a maximal monotone extension of T, and T is equal to the intersection of all its maximal monotone extensions.

The next proposition gives elementary properties of representable operators.

Proposition 2.2. *Let X be a Banach space and let $T : X \rightrightarrows X^*$ be representable. Then:*
(1) *T and T^{-1} have convex values.*
(2) *For any $n \in \mathbb{N}$, $T \cap (X \times nB_{X^*})$ is $s \times w^*$-closed in $X \times X^*$.*

Proof. (1) is well known and obvious. For (2), consider a net $\{(x_\nu, x_\nu^*)\} \subset T$, with $\{x_\nu^*\} \subset nB_{X^*}$, which $s \times w^*$-converges to (x, x^*). Clearly, x^* belongs to nB_{X^*}. It remains to show that $(x, x^*) \in T$. By Proposition 2.1 (2), $T = L(\varphi_T^*)$, so $\varphi_T^*(x_\nu, x_\nu^*) = \langle x_\nu^*, x_\nu \rangle$. Since $\{x_\nu^*\}$ is bounded, we have $\langle x_\nu^*, x_\nu \rangle \to \langle x^*, x \rangle$, hence, from the $s \times w^*$-lower semicontinuity of φ_T^* we derive that $\varphi_T^*(x, x^*) \leq \langle x^*, x \rangle$. Since $\varphi_T^* \in \mathcal{F}$, the reverse inequality also holds, so $(x, x^*) \in L(\varphi_T^*) = T$. □

A monotone operator $T : X \rightrightarrows X^*$ is said to be *maximal monotone in* $\Omega \subset X$ provided the monotone set $T \cap (\Omega \times X^*)$ is maximal (under set inclusion) in the family of all monotone sets contained in $\Omega \times X^*$. A set-valued mapping $T : Z \rightrightarrows Y$ between topological spaces Z and Y is said to be *upper semicontinuous* at $z \in Z$ provided for any open V containing Tz there is an open neighborhood U of z such that $T(U) \subset V$. When Y is a linear topological space with topology τ, an upper semicontinuous $T : Z \rightrightarrows Y$ with nonempty τ-compact convex values is called *τ-cusco*, and a τ-cusco $T : Z \rightrightarrows Y$ is called *minimal τ-cusco* if its graph does not contain the graph of any other τ-cusco between Z and Y.

The next theorem exhibits a case where representable and maximal monotone operators coincide.

Theorem 2.3. *Let X be a Banach space and let $T : X \rightrightarrows X^*$ be representable with $\Omega = \operatorname{int} \operatorname{Dom} T \neq \emptyset$. Then, T is maximal monotone in Ω. In particular, T is minimal w^*-cusco in Ω.*

Proof. We claim that T is strong-to-weak* upper semicontinuous at every point in Ω. Indeed, suppose T is not strong-to-weak* upper semicontinuous at some $x \in \Omega$. Then, there exists an w^*-open set V containing Tx and sequences $\{x_n\} \subset X$ and $\{x_n^*\} \subset X^*$ such that

$$x_n \to x \quad \text{and} \quad \forall n \in \mathbb{N},\, x_n^* \in Tx_n \setminus V. \tag{2.2}$$

Since T is locally bounded at x (see, e.g., Phelps [14, Theorem 2.28]), the sequence $\{x_n^*\}$ is bounded. Hence, it admits a bounded subnet $\{x_\alpha^*\}$ w^*-converging to a certain x^*. Thus, for some $n \in \mathbb{N}$, the net $\{(x_\alpha, x_\alpha^*)\}$ is contained in $T \cap (X \times nB_{X^*})$ and $s \times w^*$-converges to (x, x^*). From Proposition 2.2, we conclude that $(x, x^*) \in T$. On the other hand, (2.2) implies that $x_\alpha^* \notin V$ for every α, hence $x^* \notin V$. This is a contradiction because $x^* \in Tx \subset V$.

Thus, $T : \Omega \rightrightarrows X^*$ is monotone and strong-to-weak* upper semicontinuous with nonempty w^*-closed convex values (by Proposition 2.2) on the open set Ω. Therefore, T is maximal monotone and minimal w^*-cusco in Ω (see, e.g., [14, Lemma 7.7 and Theorem 7.9]). □

Let T be a non-empty monotone operator. By Proposition 2.1 (1) and (2.1), $L(\varphi_T^*)$ is a representable extension of T. Actually, $L(\varphi_T^*)$ is the smallest representable extension of T, that is, $L(\varphi_T^*)$ is equal to the intersection of all the representable extensions of T. In finite-dimensional spaces, a more precise result holds: $L(\varphi_T^*)$ is equal to the intersection of all the maximal monotone extensions of T (see Martínez-Legaz-Svaiter [10] for the details).

In the sequel, we shall use the following notation for a non-empty monotone operator T: we let $T^0 = \{\, (x, x^*) \in X \times X^* : \langle y^* - x^*, y - x\rangle \geq 0,\ \forall (y, y^*) \in T \,\}$ and $\mathcal{M}(T) = \{S \subset X \times X^* : S \text{ maximal monotone}, T \subset S\}$. Then $T^0 = \bigcup\{S : S \in \mathcal{M}(T)\}$ and $T^{00} = \bigcap\{S : S \in \mathcal{M}(T)\}$, so that $T \subset T^{00} \subset T^0$. We have:

(1) $T = T^0$ if and only if T is maximal monotone,
(2) $T = T^{00}$ if and only if T is the intersection of all its maximal monotone extensions,
(3) $T^0 = T^{00}$ if and only if T has a unique maximal monotone extension.

3. Limits of sequences of maximal monotone operators

Let $\{T_n\} \subset X \times Y$ be a sequence of operators between topological spaces (X, τ) and (Y, β). The *sequential lower limit* of $\{T_n\}$, w.r.t. the product topology $\tau \times \beta$, is the operator

$$\tau \times \beta\text{-}\liminf T_n = \{\tau \times \beta\text{-}\lim_n (x_n, y_n) : (x_n, y_n) \in T_n, \text{ for all } n \in \mathbb{N}\},$$

while its *sequential upper limit* is the operator

$$\tau \times \beta\text{-}\limsup T_n = \{\tau \times \beta\text{-}\lim_k (x_{n_k}, y_{n_k}) : (x_{n_k}, y_{n_k}) \in T_{n_k}, \text{for an infinite } \{n_k\} \subset \mathbb{N}\}.$$

Let now $\{T_n\} \subset X \times X^*$ be a sequence from a reflexive Banach space to its dual. Denote by s and w respectively the strong and weak topologies in X and X^*. Then, $\{T_n\}$ is said to *converge in the sense of Painlevé-Kuratowski* to the operator T, written $T = \text{PK-}\lim T_n$, if

$$s \times s\text{-}\limsup T_n \subset T \subset s \times s\text{-}\liminf T_n,$$

while $\{T_n\}$ is said to *converge in the sense of Mosco* to T, written $T = \text{M-}\lim T_n$, if

$$w \times w\text{-}\limsup T_n \subset T \subset s \times s\text{-}\liminf T_n.$$

In the sequel, we simply write $\liminf T_n$ instead of $s \times s\text{-}\liminf T_n$.

From now on, we assume that X is a reflexive Banach space, with a strictly convex norm in X and a strictly convex dual norm in X^* (always possible thanks to Asplund [1]). With such norms, the *duality mapping* $J : X \rightrightarrows X^*$ given by

$$Jx := \left\{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\right\}$$

is single-valued, bijective and maximal monotone, and it is well known (see, e.g., Rockafellar [18]) that a monotone operator $T : X \rightrightarrows X^*$ is maximal monotone if and only if the operator $J + T$ is surjective. Moreover, in that case, the operator $(J + T)^{-1}$ is single-valued on X^*. According to this result, to any maximal monotone $T : X \rightrightarrows X^*$ and $\lambda > 0$, we associate its *resolvent* $J_\lambda^T : X \to X$ which to each $x \in X$ assigns the unique solution $x_\lambda \in \operatorname{Dom} T$ of the inclusion $0 \in J(x_\lambda - x) + \lambda T x_\lambda$, and its *Yosida regularization* $T_\lambda : X \rightrightarrows X^*$ given by $T_\lambda x = J(x - x_\lambda)/\lambda$. Then, T_λ is a single-valued maximal monotone operator of all of X into X^*, and satisfies $T_\lambda x \in T x_\lambda$.

Let now $\{T_n : X \rightrightarrows X^*\}$ be a sequence of maximal monotone operators. For $(x, x^*) \in X \times X^*$ and $n \in \mathbb{N}$, we consider the unique solution $x_n = J_{T_n}(x, x^*)$ of the inclusion

$$x^* \in J(x_n - x) + T_n(x_n). \tag{3.1}$$

Lemma 3.1. *Let X be reflexive strictly convex with a strictly convex dual X^* and let $\{T_n : X \rightrightarrows X^*\}$ be a sequence of maximal monotone operators. Set $T = \liminf T_n$ and assume $\operatorname{Dom} T \neq \emptyset$. Then, for any $(x, x^*) \in X \times X^*$, the sequence $\{x_n\}$ of solutions of (3.1) is bounded, and for any subsequence $\{x_{n_k}\}$ with $x_{n_k} \rightharpoonup \overline{x}$ and $J(x_{n_k} - x) \rightharpoonup \eta^*$, we have*

$$\langle x^* - \eta^* - y^*, \overline{x} - y\rangle + \langle \eta^*, \overline{x} - x\rangle \geq \limsup \|x_{n_k} - x\|^2, \quad \forall (y, y^*) \in T. \tag{3.2}$$

In particular, $(\overline{x}, x^* - \eta^*) \in T^0$.

Proof. Let $(y, y^*) \in T$. By definition of T, there exists a sequence $\{(y_n, y_n^*)\}$ such that $(y_n, y_n^*) \to (y, y^*)$ and $(y_n, y_n^*) \in T_n$ for each $n \in \mathbb{N}$. Using the monotonicity of T_n and (3.1) we get

$$\big\langle x^* - J(x_n - x) - y_n^*, x_n - y_n \big\rangle \geq 0,$$

hence

$$\langle x^* - y_n^*, x_n - y_n\rangle \geq \|x_n - x\|^2 + \langle J(x_n - x), x - y_n\rangle. \tag{3.3}$$

Since the sequences $\{y_n\}$ and $\{y_n^*\}$ are bounded, from (3.3) we deduce that $\{x_n\}$ is bounded.

Let $\{x_{n_k}\}$ be a subsequence such that $x_{n_k} \rightharpoonup \overline{x}$ and $J(x_{n_k} - x) \rightharpoonup \eta^*$. Taking the limit in (3.3) we obtain

$$\langle x^* - y^*, \overline{x} - y\rangle \geq \limsup \|x_{n_k} - x\|^2 + \langle \eta^*, x - y\rangle,$$

which clearly gives (3.2).

On the other hand, the monotonicity of J implies

$$\Big\langle J(x_{n_k} - x) - J(\overline{x} - x), (x_{n_k} - x) - (\overline{x} - x)\Big\rangle \geq 0,$$

that is,

$$\|x_{n_k} - x\|^2 \geq \Big\langle J(x_{n_k} - x), \overline{x} - x\Big\rangle - \Big\langle J(\overline{x} - x), x_n - \overline{x}\Big\rangle,$$

so passing to the limit we get

$$\liminf \|x_{n_k} - x\|^2 \geq \langle \eta^*, \overline{x} - x\rangle.$$

Combining this inequality with (3.2) we derive that

$$\langle x^* - \eta^* - y^*, \overline{x} - y\rangle \geq 0, \quad \forall (y, y^*) \in T,$$

proving that $(\overline{x}, x^* - \eta^*) \in T^0$. □

The following proposition provides a convenient description of the set $\liminf T_n$ in terms of limits of sequences of solutions of (3.1).

Proposition 3.2. *Let X be reflexive strictly convex with a strictly convex dual X^*. Let $\{T_n : X \rightrightarrows X^*\}$ be a sequence of maximal monotone operators. Then, $(x, x^*) \in \liminf T_n$ if and only if $x = \lim x_n$, where $\{x_n\}$ is the sequence of solutions of equation (3.1).*

Proof. Let $(x, x^*) \in \liminf T_n$. Consider the sequence $\{x_n\}$ of solutions of (3.1). By Lemma 3.1, any subsequence of $\{x_n\}$ admits a converging subsequence $\{x_{n_k}\}$ which satisfies (3.2). Using (3.2) with $(y, y^*) = (x, x^*)$, we get $0 \geq \limsup \|x_{n_k} - x\|^2$, that is, $x_{n_k} \to x$. We conclude that the whole sequence $\{x_n\}$ converges to x.

Conversely, by (3.1), $(x_n, x^* - J(x_n - x)) \in T_n$ for every $n \in \mathbb{N}$. If $x_n \to x$, then $x^* - J(x_n - x) \to x^*$, so $(x, x^*) \in \liminf T_n$ by definition of $\liminf T_n$. □

We now turn to our main result in this section.

Theorem 3.3. *Let X be reflexive strictly convex with a strictly convex dual X^* and let $\{T_n : X \rightrightarrows X^*\}$ be a sequence of maximal monotone operators.*

(1) *$T = \liminf T_n$ is representable;*

(2) *If the duality mapping J is weakly sequentially continuous (as when X is a Hilbert space or ℓ_p, $1 < p < \infty$) and $T = w \times w\text{-}\liminf T_n$, then T is the intersection of all its maximal monotone extensions.*

Proof. (1) Without loss of generality, we may assume that T is nonempty (the empty set is representable). It is well known and easily seen that T is monotone, so, by Proposition 2.1, proving that T is representable amounts to proving that $L(\varphi_T^*) \subset T$. According to Lemma 3.1, for any $(x, x^*) \in X \times X^*$, the sequence $\{x_n\}$ of solutions of (3.1) is bounded and any subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup \overline{x}$ and $J(x_{n_k} - x) \rightharpoonup \eta^*$ satisfies (3.2). By definition of φ_T, (3.2) can be rewritten as

$$\langle x^* - \eta^*, \overline{x}\rangle + \langle \eta^*, \overline{x} - x\rangle \geq \limsup \|x_{n_k} - x\|^2 + \varphi_T(\overline{x}, x^* - \eta^*). \tag{3.4}$$

Since

$$\begin{aligned} \varphi_T^*(x, x^*) &= \sup\{\, \langle (y, y^*), (x, x^*)\rangle - \varphi_T(y, y^*) : (y, y^*) \in X \times X^* \,\} \\ &\geq \langle (\overline{x}, x^* - \eta^*), (x, x^*)\rangle - \varphi_T(\overline{x}, x^* - \eta^*) \\ &= \langle x^*, \overline{x}\rangle + \langle x^* - \eta^*, x\rangle - \varphi_T(\overline{x}, x^* - \eta^*), \end{aligned}$$

we derive from (3.4) that

$$\varphi_T^*(x, x^*) \geq \limsup \|x_{n_k} - x\|^2 + \langle x^*, x\rangle. \tag{3.5}$$

Now, let $(x, x^*) \in L(\varphi_T^*)$. Then (3.5) reduces to

$$0 \geq \limsup \|x_{n_k} - x\|^2,$$

showing that $x_{n_k} \to x$. It follows that $x_n \to x$, since this subsequence was taken arbitrarily. From Proposition 3.2 we conclude that $(x, x^*) \in T$, as required.

(2) Again, without loss of generality, we may assume that T is nonempty (the empty set is equal to the intersection of all the maximal monotone sets). Assuming J weakly sequentially continuous and $T = w \times w\text{-}\liminf T_n$, we show that $T^{00} = T$. By Lemma 3.1, for any $(x, x^*) \in X \times X^*$, the sequence $\{x_n\}$ of solutions of (3.1) is bounded and if $\{x_{n_k}\}$ is any subsequence such that $x_{n_k} \rightharpoonup \overline{x}$ and $J(x_{n_k} - x) \rightharpoonup \eta^*$, then $(\overline{x}, x^* - \eta^*) \in T^0$. In fact $\eta^* = J(\overline{x} - x)$ since J is weakly sequentially continuous, so $(\overline{x}, x^* - J(\overline{x} - x)) \in T^0$. Now, let $(x, x^*) \in T^{00}$. Then

$$\big\langle x^* - J(\overline{x} - x) - x^*, \overline{x} - x \big\rangle \geq 0,$$

that is, $-\|\overline{x} - x\|^2 \geq 0$. Hence, $\overline{x} = x$. We derive that $x_n \rightharpoonup x$ and $x^* - J(x_n - x) \rightharpoonup x^*$. Since

$$\big(x_n, x^* - J(x_n - x)\big) \in T_n, \quad \forall n \in \mathbb{N},$$

we have $(x, x^*) \in w \times w\text{-}\liminf T_n$, therefore, by assumption, $(x, x^*) \in T$. This shows that $T^{00} \subset T$, which was to be proved. □

Corollary 3.4. *Let X be a finite dimensional space and let $\{T_n : X \rightrightarrows X\}$ be a sequence of maximal monotone operators. Then, $\liminf T_n$ is the intersection of all its maximal monotone extensions.*

Proof. In finite dimensional spaces, strong and weak topologies coincide. The result therefore follows directly from Assertion (2) in the theorem. □

Example. In $\mathbb{R}\times\mathbb{R}$, let $T_n=\{0\}\times\mathbb{R}$ for even n, $T_n=\mathbb{R}\times\{0\}$ for odd n. Then:

(1) Each T_n is maximal monotone,
(2) $\liminf T_n=\{(0,0)\}$ is equal to the intersection of all its maximal monotone extensions, but is not maximal monotone,
(3) $\limsup T_n=(\{0\}\times\mathbb{R})\cup(\mathbb{R}\times\{0\})$ is not even monotone.

The next result extends Attouch's theorem (see [2, 3]) asserting that *the class of maximal monotone operators on a finite dimensional space is closed with respect to Painlevé-Kuratowski convergence.*

Theorem 3.5. *Let X be reflexive strictly convex with a strictly convex dual X^* and let $\{T_n : X \rightrightarrows X^*\}$ be a sequence of maximal monotone operators. Then, when nonempty, the Mosco limit $M\text{-}\lim T_n$ is maximal monotone.*

Proof. Assume that $T=s\times s\text{-}\liminf T_n=w\times w\text{-}\limsup T_n$, with $\operatorname{Dom}T\neq\emptyset$. We show that $T^0=T$. Invoking again Lemma 3.1, we know that, for any $(x,x^*)\in X\times X^*$, the sequence $\{x_n\}$ of solutions of (3.1) is bounded and any subsequence $\{x_{n_k}\}$ such that $x_{n_k}\rightharpoonup\overline{x}$ and $J(x_{n_k}-x)\rightharpoonup\eta^*$ satisfies (3.2). Since

$$\big(x_{n_k},x^*-J(x_{n_k}-x)\big)\in T_{n_k},\quad \forall k\in\mathbb{N},$$

we have $(\overline{x},x^*-\eta^*)\in w\times w\text{-}\limsup T_n$, hence, $(\overline{x},x^*-\eta^*)\in T$. Using (3.2) with $(y,y^*)=(\overline{x},x^*-\eta^*)$, we get

$$\langle\eta^*,\overline{x}-x\rangle\geq\limsup\|x_{n_k}-x\|^2. \tag{3.6}$$

Now, let $(x,x^*)\in T^0$. We have

$$\langle\eta^*,x-\overline{x}\rangle=\big\langle x^*-(x^*-\eta^*),x-\overline{x}\big\rangle\geq 0,$$

so (3.6) yields

$$0\geq\limsup\|x_{n_k}-x\|^2.$$

As in the proof of Assertion (1) in Theorem 3.5, we conclude that $(x,x^*)\in T$, as required. □

4. Application 1: the variational sum

Again in this section, we are given a reflexive Banach space X, with strictly convex norms in X and X^*, so that the duality mapping $J:X\to X^*$ is single-valued, bijective and maximal monotone.

Let $T_1,T_2:X\rightrightarrows X^*$ be maximal monotone. It is well known that the point-wise Minkowski sum T_1+T_2 is a monotone operator, which is not maximal in general, not even representable. This is the reason why in recent years different ways of summing two maximal monotone operators were considered in order to have more chances to get maximality or at least representability. We refer to Revalski [15] for a survey on generalized sums and compositions.

One such concept is the so-called *variational sum*, introduced by Attouch-Baillon-Théra [3] in Hilbert spaces, then generalized to reflexive Banach spaces by Revalski-Théra [16, 17]. It is defined as follows:

$$T_1\underset{v}{+}T_2:=\bigcap_{\mathcal{I}}\liminf_n\,(T_{1,\lambda_n}+T_{2,\mu_n}),$$

where $\mathcal{I}=\Big\{\{(\lambda_n,\mu_n)\}\subset\mathbb{R}^2:\lambda_n,\,\mu_n\geq 0,\ \lambda_n+\mu_n>0,\lambda_n,\mu_n\to 0\Big\}$, and T_{1,λ_n}, T_{2,μ_n} are the Yosida approximations of T_1 and T_2 respectively. [The original definition uses topological non-sequential limit, but it is easily seen that both definitions are equivalent.]

In this setting, the variational sum turns out to be bigger than the point-wise sum:

$$T_1 + T_2 \subset T_1 \underset{v}{+} T_2. \tag{4.1}$$

This fact was recently established by García [8, Corollary 3.7]. Also, it has been shown that the variational sum of two subdifferentials of lower semicontinuous convex functions is the subdifferential of the sum of the two functions, hence a maximal monotone operator (see [3, 17]). But the general question whether the variational sum of two arbitrary maximal monotone operators is maximal monotone is still open. However, we have the following:

Theorem 4.1. *Let X be reflexive strictly convex with a strictly convex dual X^* and let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone. Assume* $\operatorname{Dom}(T_1 + T_2) \neq \emptyset$. *Then, the variational sum $T_1 \underset{v}{+} T_2$ is a representable extension of the point-wise sum $T_1 + T_2$.*

Proof. For every $\{(\lambda_n, \mu_n)\} \in \mathcal{I}$, the operators $T_n = T_{1,\lambda_n} + T_{2,\mu_n}$, $n \in \mathbb{N}$, are maximal monotone. Indeed, for every $n \in \mathbb{N}$, at least one of the parameters λ_n or μ_n is different from 0, so at least one of the operators T_{1,λ_n} or T_{2,μ_n} is single-valued and maximal monotone, hence their sum is maximal monotone. On the other hand, by (4.1),

$$T_1 + T_2 \subset T_1 \underset{v}{+} T_2 \subset \liminf\,(T_{1,\lambda_n} + T_{2,\mu_n}),$$

hence $\operatorname{Dom}(\liminf T_n) \neq \emptyset$. Therefore, according to Theorem 3.3, for every $\{(\lambda_n, \mu_n)\} \in \mathcal{I}$, the operator

$$\liminf\,(T_{1,\lambda_n} + T_{2,\mu_n})$$

is representable. Since the intersection of representable operators is a representable operator, we conclude that $T_1 \underset{v}{+} T_2$ is representable. □

Example. (See García-Lassonde-Revalski [9, Example 3.11] and García [8, Example 3.13].) Let $X = \ell_2 \times \ell_2$ and identify X^* with X. Let $\operatorname{Dom} T := D \times D$ with

$$D := \{\{x_n\} \subset \ell_2 : \{2^n x_n\} \in \ell_2\},$$

and let $T : \operatorname{Dom} T \to X$ be defined by

$$T(\{x_n\}, \{y_n\}) := (\{2^n y_n\}, -\{2^n x_n\}).$$

Consider $T_1 := T$ and $T_2 := -T$. These operators are linear, anti-symmetric and maximal monotone with common dense domain $\operatorname{Dom} T$. Clearly, $T_1 + T_2$ *is not representable*, because $T_1 + T_2$ is not closed in $X \times X^*$, since $T_1 + T_2 \equiv 0$ with $\operatorname{Dom}(T_1 + T_2) = \operatorname{Dom} T$ proper dense subset of X, while $T_1 \underset{v}{+} T_2$ *is representable* (in fact maximal monotone), since $(T_1 \underset{v}{+} T_2) \equiv 0$ with $\operatorname{Dom}(T_1 \underset{v}{+} T_2) = X$.

5. Application 2: the variational composition

Let X and Y be reflexive Banach spaces, supplied with strictly convex norms as well as their dual spaces, let $T : X \rightrightarrows X^*$ be maximal monotone and let $A : Y \to X$ be linear continuous with adjoint $A^* : X^* \to Y^*$. The usual point-wise composition $A^*TA : Y \rightrightarrows Y^*$ is given by

$$A^*TAy = \{y^* = A^*x^* : x^* \in TAx\}.$$

As in the case of the point-wise sum, this operator is monotone but in general not even representable, whence the idea to consider different types of compositions. Based on the same

idea as the variational sum, the so-called *variational composition*, introduced by Pennanen-Revalski-Théra [11], is defined by:

$$(A^*TA)_v = \bigcap_{\mathcal{J}} \liminf A^*T_{\lambda_n}A,$$

where $\mathcal{J} = \Big\{\{\lambda_n\} \subset \mathbb{R} : \lambda_n > 0,\ \lambda_n \searrow 0\Big\}$, and T_{λ_n} is the Yosida approximation of T. [Again the original definition uses topological non-sequential limit, but it is easily seen that both definitions are equivalent.]

It is well known that the point-wise composition can be expressed as a point-wise sum of operators. More specifically, let $N_A, \widetilde{T} : Y \times X \rightrightarrows Y \times X$ be defined by

$$\widetilde{T}(y,x) := \{0\} \times Tx, \quad \forall (y,x) \in Y \times X, \qquad N_A := \partial\delta_A,$$

where δ_A is the indicator function of the graph of A, therefore,

$$N_A(y,x) = \begin{cases} \big\{(A^*x^*, -x^*) : x^* \in X^*\big\} & \text{if } (y,x) \in A, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then

$$y^* \in A^*TAy \Leftrightarrow (y^*,0) \in (\widetilde{T} + N_A)(y, Ay). \tag{5.1}$$

Similarly, it turns out that the variational composition can be expressed via an asymmetric variant of the variational sum. Given maximal monotone operators $T_1, T_2 : X \rightrightarrows X^*$, consider the following *left variational sum*:

$$l\text{-}(T_1 \underset{v}{+} T_2) := \bigcap_{\mathcal{J}} \liminf\, (T_{1,\lambda_n} + T_2).$$

Clearly, this operator contains the variational sum,

$$T_1 \underset{v}{+} T_2 \subset l\text{-}(T_1 \underset{v}{+} T_2), \tag{5.2}$$

and in general the inclusion is proper, as the following example shows.

Example. Let $X = \mathbb{R}$, $T_1 = \partial\delta_{\{-1\}}$ and $T_2 = \partial\delta_{\{1\}}$. We have $\mathrm{Dom}\,(T_1) \cap \mathrm{Dom}\,(T_2) = \emptyset$, and one verifies that the usual and the variational sum are the trivial empty operator. On the other hand, for $\lambda > 0$ we have $T_{1,\lambda}(x) = (x+1)/\lambda$ for every $x \in \mathbb{R}$, and it can be checked that $l\text{-}(T_1 \underset{v}{+} T_2) = \{1\} \times \mathbb{R}$.

Here is the promised representation of the variational composition as a left variational sum:

Proposition 5.1. *Let X and Y be Banach spaces. Let $T : X \rightrightarrows X^*$ be maximal monotone and let $A : Y \to X$ be linear continuous. We have the following equivalence:*

$$y^* \in (A^*TA)_v(y) \Leftrightarrow (y^*,0) \in l\text{-}(\widetilde{T} \underset{v}{+} N_A)(y, Ay). \tag{5.3}$$

Proof. Notice that for every $\lambda > 0$ and $(y,x) \in Y \times X$ we have $\widetilde{T}_\lambda(y,x) = \{0\} \times T_\lambda(x)$.

Let $(y,y^*) \in (A^*TA)_v$. By definition, for any $\{\lambda_n\} \in \mathcal{J}$, there exists $\{y_n\} \subset X$ such that $(y_n, A^*T_{\lambda_n}Ay_n) \to (y, y^*)$. Observe that $(A^*T_{\lambda_n}Ay_n, -T_{\lambda_n}Ay_n) \in N_A(y_n, Ay_n)$, hence

$$(A^*T_{\lambda_n}Ay_n, 0) = (0, T_{\lambda_n}Ay_n) + (A^*T_{\lambda_n}Ay_n, -T_{\lambda_n}Ay_n) \in (\widetilde{T}_{\lambda_n} + N_A)(y_n, Ay_n).$$

Since $(A^*T_{\lambda_n}Ay_n, 0) \to (y^*, 0)$ and $(y_n, Ay_n) \to (y, Ay)$, by definition of $\liminf$ we get

$$((y^*,0),(y,Ay)) \in \liminf\,(\widetilde{T}_{\lambda_n} + N_A).$$

Since this is valid for any $\{\lambda_n\} \in \mathcal{J}$, we conclude that $(y^*, 0) \in l\text{-}(\widetilde{T} \underset{v}{+} N_A)(y, Ay)$.

Conversely, let $(y^*, 0) \in l\text{-}(\widetilde{T} \underset{v}{+} N_A)(y, Ay)$. Then, for any $\{\lambda_n\} \in \mathcal{J}$, there exists $\{(y_n, x_n)\} \subset Y \times X$ and $\{(y_n^*, x_n^*)\} \subset Y^* \times X^*$ with $(y_n^*, x_n^*) \in (\widetilde{T}_{\lambda_n} + N_A)(y_n, x_n)$ such that $(y_n, x_n) \to (y, Ay)$ and $(y_n^*, x_n^*) \to (y^*, 0)$. From the definition of N_A, we get $x_n = Ay_n$ and from

$$(y_n^*, x_n^*) = (0, T_{\lambda_n} Ay_n) + (y_n^*, x_n^* - T_{\lambda_n} Ay_n) \in \widetilde{T}_{\lambda_n} + N_A$$

we get $y_n^* = A^*(T_{\lambda_n} Ay_n - x_n^*) = A^* T_{\lambda_n} Ay_n - A^* x_n^*$. Thus, $(y_n, y_n^* + A^* x_n^*) \in A^* T_{\lambda_n} A$ and since $(y_n, y_n^* + A^* x_n^*) \to (y, y^*)$, we have

$$(y, y^*) \in \liminf A^* T_{\lambda_n} A.$$

We conclude that $y^* \in (A^*TA)_v(y)$. □

We are now ready to prove the analog of Theorem 4.1.

Theorem 5.2. *Let X and Y be reflexive strictly convex with strictly convex duals. Let $T : X \rightrightarrows X^*$ be maximal monotone and let $A : Y \to X$ be linear continuous. Assume $\operatorname{Dom} A^*TA \neq \emptyset$. Then, the variational composition $(A^*TA)_v$ is a representable extension of the point-wise composition A^*TA.*

Proof. First we show that $(A^*TA)_v$ contains A^*TA. Let $(y, y^*) \in A^*TA$. By (5.1), $(y^*, 0) \in (\widetilde{T} + N_A)(y, Ay)$. Observe that $Y \times X$ supplied with the square norm on the product is a reflexive Banach space with a strictly convex norm as well as its dual. So we may apply (4.1) with the maximal monotone mappings $\widetilde{T}$ and N_A acting on $Y \times X$ to obtain $(y^*, 0) \in (\widetilde{T} \underset{v}{+} N_A)(y, Ay)$, hence, by (5.2), $(y^*, 0) \in l\text{-}(\widetilde{T} \underset{v}{+} N_A)(y, Ay)$. This amounts to $(y, y^*) \in (A^*TA)_v$ according to Proposition 5.1.

Next, we show that $(A^*TA)_v$ is representable. This follows in the same way as for the variational sum (Theorem 4.1). Indeed, for any $\lambda_n > 0$ the operator $A^* T_{\lambda_n} A : Y \to Y$ is single-valued everywhere defined maximal monotone because so is T_{λ_n} (see, e.g., [19]). Moreover $\operatorname{Dom}(\liminf A^* T_{\lambda_n} A) \neq \emptyset$ because $\operatorname{Dom} A^*TA \neq \emptyset$ and $A^*TA \subset (A^*TA)_v \subset \liminf A^* T_{\lambda_n} A$. Hence, in view of Theorem 3.3, for every $\{\lambda_n\} \in \mathcal{J}$, the operator $\liminf A^* T_{\lambda_n} A$ is representable, so also is $(A^*TA)_v$ as intersection of such representable operators. □

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